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LETTER TO THE EDITOR

A self-dual point for triangular Ashkin–Teller models

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Abstract. It is shown that there is a unique point at which high temperature expressions for a special class of Ashkin–Teller models on the triangular lattice transform into low temperature expressions for the same models. It is suggested that this point is the critical point.

The Ashkin–Teller model that we consider is equivalent to two parallel Ising lattices with spin variables σ_i, $S_i = \pm 1$ and interactions $-J\sigma_i\sigma_j$, $-JS_iS_j$. They are coupled by a four-spin interaction $-J_4\sigma_i\sigma_jS_iS_j$. The original model (Ashkin and Teller 1943) allowed different interactions in each subsystem, in which case Wu and Lin (1974) have suggested there should be two phase transitions. We consider only the case of equal interactions in each layer. For the square lattice system Wegner (1972) has shown that the critical temperature is equal to that of the eight-vertex model of the same J, J_4 values. In the eight-vertex model the two square sublattices are staggered rather than having their sites corresponding as in the Ashkin–Teller model.

For the triangular lattice Ashkin–Teller model there are two special cases for which the critical point is known. $J_4 = 0$ is two independent Ising models or equivalently a four-state planar Potts model (Betts 1964, Suzuki 1967). $J_4 = 1$ is the standard Potts model whose critical point was found by Kim and Joseph (1974).

To discuss transformations of the Ashkin–Teller model we treat it in its original form as a four-state system with states indexed 0 to 3 corresponding to (σ, S) configurations $(+, +), (+, -), (-, -), (-, +)$. (All indices take values 0 to 3.)

The interaction energies are

$$J_{jj} = 0 \tag{1}$$

$$J_{01} = J_{12} = J_{23} = J_{03} = 2J + 2J_4 \tag{2}$$

$$J_{02} = J_{13} = 4J \tag{3}$$

having chosen the fully aligned state to be the zero of energy.

The possibility of a general high temperature expression is suggested by the result of Domb (1974) that it is possible to diagonalize the transfer matrix, $\boldsymbol{V}$, for both the special cases $J_4 = 0, J_4 = 1$ using the *same* matrix, $\boldsymbol{T}$, whose elements are

$$T_{jk} = \tfrac{1}{2}\mathrm{i}^{jk} \qquad j, k = 0, 3, \qquad \mathrm{i} = \sqrt{-1} \tag{4}$$

and in fact $\boldsymbol{T}$ does diagonalize the transfer matrix for the general case, the eigenvalues being

$$\lambda_0 = 1 + 2x + y \tag{5}$$

$$\lambda_1 = \lambda_3 = 1 - y \tag{6}$$

$$\lambda_2 = 1-2x+y \tag{7}$$

$$x = \exp -2\beta(J+J_4) \tag{8}$$

$$y = \exp -4\beta J. \tag{9}$$

For the three-state Potts model, Mittag and Stephen (1971) give a graphical expansion for the partition function. The appropriate generalization of their vertex rules is suggested by the work of Domb (1974, equations (44), (45)).

To construct a high temperature expression for the partition function we assign an arbitrary direction to each bond and for each bond from a site in state j to a site in state k, we include in the partition function a term V_{jk} in the form $\Sigma_m T_{jm}\lambda_m \overline{T}_{mk}$. The directions are arbitrary and merely reflect the fact that we have introduced a direction dependence into the formalism by making a choice between $\boldsymbol{T\Lambda\overline{T}}$ and $\boldsymbol{\overline{T}\Lambda T}$ to express V.

At any particular site in state j we have

$$\prod_{\substack{\text{arrows}\\ \text{out}}} T_{jn} \prod_{\substack{\text{arrows}\\ \text{in}}} \overline{T}_{n'j} = (\tfrac{1}{2})^z \, \mathrm{i}^{j(\Sigma_1 n - \Sigma_2 n')} \tag{10}$$

and summing over all configurations of the site, ie over all four values of j, gives zero unless

$$\sum_1 n - \sum_2 n' = 0 \text{ (modulo 4)} \tag{11}$$

where $\Sigma_1 n$ is a sum over n values for all the 'out' bonds and $\Sigma_2 n'$ is a sum over all the 'in' bonds. z is the coordination number. The high temperature expansion thus takes the form (for N sites)

$$Z = \lambda_0^{Nz/2} 2^{-Nz} 4^N G(1, \lambda_1/\lambda_0, \lambda_2/\lambda_0, \lambda_3/\lambda_0) \tag{12}$$

where G is a sum over all configurations of types of bonds on the lattice that satisfy (11) and have bonds of type j contributing λ_j/λ_0.

We now relate the G functions for honeycomb (hc) and triangular (tri) lattices. We consider a site a on the honeycomb lattice with its three neighbours b, c, d and relate all allowed configurations to configurations on the triangle bcd. To preserve the relationship of these sites to the rest of the lattice we must equate terms that have the same set of 'currents' (α, β, γ) into sites b, c, d from outside the sites a, b, c, d.

currents	hc contributions	sum of tri contribution
$(1, 0, -1)$	v^2	$r+r^2+r^2u+u^2r$
$(2, 0, 2)$	w^2	$u+u^2+2r^3$
$(0, 0, 0)$	1	$1+2r^3+2u^3$
$(1, 1, 2)$	v^2w	$r^2+2ur+r^2u$

We can put

$$G_{\text{hc}}(1, v, w, v) = (1+2r^3+u^3)^{-M} G_{\text{tri}}(1, r, u, r) \tag{13}$$

where M is the number of triangular lattice sites.

$$v^2 = (r+r^2+r^2u+u^2r)/(1+2r^3+2u^3) \tag{14}$$

$$w^2 = (u+u^2+2r^3)/(1+2r^3+2u^3) \tag{15}$$

and for consistency we require

$$(r^2+2ur+r^2u)^2(1+2r^3+u^3) = (r+r^2+r^2u+ru^2)^2(u+u^2+2r^3). \tag{16}$$

The existence of a high temperature to low temperature duality transformation is known for the two special cases $J_4 = 0$, $J_4 = 1$ and Wegner (1973) has shown that constraints such as (11) can be regarded as the generators of such transformations. The work of Mittag and Stephen (1971) (see especially figure 2) suggests how such a transformation may be constructed in a graphical form. Going from a low temperature configuration to a high temperature configuration the exponentials of the energies (1), (2) and (3) map onto 1, λ_1/λ_0 or λ_3/λ_0 and λ_2/λ_0. The arrows are needed only to decide between λ_1 and λ_3. We order the states 0, 1, 2, 3, 0, 1, 2 essentially as done by Mittag and Stephen and use λ_1/λ_0 if the Mittag and Stephen convention gives an arrow parallel to our original bond arrow, and use λ_3/λ_0 otherwise. (This topological form of the duality transformation was considered by Kihara *et al* (1954).)

We have

$$Z_{\text{tri}}(x, y) = G_{\text{hc}}(1, x, y, x). \tag{17}$$

To combine equations (12), (13) and (17) to find a self-dual point for the triangular lattice we must first satisfy equation (16). We then have to have low temperature variables transforming into high temperature variables $r = \lambda_1/\lambda_0$, $u = \lambda_2/\lambda_0$, so that

$$x^2 = \left(\frac{1-u}{1+2r+u}\right) = \frac{r+r^2+r^2u+ru^2}{1+2r^3+u^3} \tag{18}$$

$$y^2 = \left(\frac{1-2r+u}{1+2r+u}\right)^2 = \frac{u+u^2+2r^3}{1+2r^3+u^3}. \tag{19}$$

For any particular model we fix J_4/J so that there is only one independent variable J/kT. While equations (18), (19) can give a transformation of a high temperature Ashkin–Teller system into a low temperature Ashkin–Teller system, the equations can only represent a transformation of a system into itself if they can both be satisfied simultaneously.

Assuming for the moment that a solution exists we have

$$Z_{\text{tri}}(x, y) = (1+2r^3+u^3)^{-N}\lambda_0^{-N/2}2^{4N}Z_{\text{tri}}(r, u). \tag{20}$$

For a self-dual point we must equate the initial factors to 1. This reduces to

$$(1+u+2r)^3 = 4(1+2r^3+u^3) \tag{21}$$

so for a self-dual point we must satisfy equations (16), (18), (19) and (21) simultaneously, remembering that for fixed J_4/J, any *one* of the four equations will determine T (except that equation (16) is true for all T if $J_4 = 0$).

In fact they can all be satisfied and so there is a self-dual point for which the simplest equation is the reduced form of equation (21).

$$4r^2+2(1+u)r-(1-u)^2 = 0 \tag{22}$$

$$r = (1-y)/(1+2x+y) \tag{23}$$

$$u = (1-2x+y)/(1+2x+y) \tag{24}$$

whence

$$1-y-2x^2 = 0. \tag{25}$$

The solutions of equation (22) include the two known solutions. The remaining possible check is to calculate dT_c/dJ_4 at $J_4 = 0$ both from equation (22) and from the perturbation approach of Kadanoff and Wegner (1971).

Both methods agree, giving

$$\frac{d\beta_c}{d(J_4/J)} = -\tfrac{2}{3}\beta_c \qquad (\text{at } J_4 = 0). \tag{26}$$

Since we have a solution that gives the critical point in all known cases it seems probable that it is the critical point in the general case. For the $J_4 = 1$ case Kim and Joseph were able to include an auxiliary interaction so that the star triangle transformation remained valid. In the present case we would need a more complicated interaction and would still have to contend with the fact that away from the self-dual point the transformation would change the J_4/J ratio and so we would have a transformation connecting different models rather than connecting high and low temperature regions of one model.

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